THE KURATOWSKI COVERING CONJECTURE FOR GRAPHS OF ORDER < 10 FOR THE NONORIENTABLE SURFACES OF GENUS 3 AND 4

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Abstract. Kuratowski proved that a finite graph embeds in the plane if it does not contain a subdivision of either K_5 or $K_{3,3}$, called Kuratowski subgraphs. A conjectured generalization of this result to all nonorientable surfaces says that a finite minimal forbidden subgraph for the nonorientable surface of genus $\tilde{g}$ can be written as the union of $\tilde{g}+1$ Kuratowski subgraphs such that the union of each pair of these fails to embed in the projective plane, the union of each triple of these fails to embed in the Klein bottle if $\tilde{g} \geq 2$, and the union of each triple of these fails to embed in the torus if $\tilde{g} \geq 3$. We show that this conjecture is true for all minimal forbidden subgraphs of order < 10 for the nonorientable surfaces of genus 3 and 4.

1. Introduction

We use the same terminology as in [12] unless otherwise specified. $G_1 \vee G_2$ denotes the graph obtained by identifying one vertex of G_1 and one vertex of G_2.

Kuratowski [11] showed that minimal forbidden subgraphs for the plane are K_5 and $K_{3,3}$. Given a graph G, any subgraph of G that is a subdivision of K_5 or $K_{3,3}$ is called a *Kuratowski subgraph* of G.

Then one might ask if Kuratowski's result can be extended to higher genus surfaces in terms of Kuratowski subgraphs. Glover has conjectured that if a finite graph G is a minimal forbidden subgraph for the nonorientable surface $\mathbb{N}_{\tilde{g}}$, then G can be written as the union of $\tilde{g}+1$ Kuratowski subgraphs such that the union of each pair of these fails to embed in the projective plane, the union of each triple of these fails to embed in the Klein bottle if $\tilde{g} \geq 2$, and the union of each triple of these fails to embed in the torus if $\tilde{g} \geq 3$. It should be noted that G is the union of $\tilde{g}+1$ Kuratowski subgraphs, i.e., every edge in G is an edge in at least one of the Kuratowski subgraphs. The set of $\tilde{g}+1$ subgraphs described in the conjecture is called a *Kuratowski covering* and the conjecture is called the *Kuratowski covering conjecture*. In this paper, we prove the following restricted version of the above conjecture.

Theorem 1.1. *The Kuratowski covering conjecture is true for every graph of order < 10 for $\mathbb{N}_3$ and $\mathbb{N}_4$.*

We prove this theorem by providing a Kuratowski covering for every minimal forbidden subgraph of order < 10 for $\mathbb{N}_3$ and $\mathbb{N}_4$.

We use the complete lists of minimal forbidden subgraphs of order < 10 for some surfaces. Archdeacon and Huneke [2] showed that there are finitely many minimal forbidden subgraphs for nonorientable surfaces, and Robertson and Seymour [15] independently showed that there are finitely many minimal forbidden subgraphs for

arbitrary surfaces. Kuratowski [11] showed that minimal forbidden subgraphs for the plane are K_5 and $K_{3,3}$. A list of minimal forbidden subgraphs for the projective plane has been found by Glover, Huneke, and Wang [5] and Archdeacon [1] proved that this list is complete. However, for the higher genus surfaces, the complete lists of minimal forbidden subgraphs are not known. The complete list of 8-vertex minimal forbidden subgraphs for the Klein bottle has been found by Huneke, McQuillan, and Richter [7] and the complete list of 9-vertex minimal forbidden subgraphs for the Klein bottle have been found by Cashy [3] and Hur [10]. The complete list of 8-vertex minimal forbidden subgraphs for the torus has been found by Duke and Haggard [4], and the complete list of 9-vertex minimal forbidden subgraphs for the torus has been found by Hlavacek [6]. We note that Wendy Myrvold in the Department of Computer Science, University of Victoria independently found about 235,000 minimal forbidden subgraphs for the torus using a computer [13], [14]. In particular, the list of 9-vertex minimal forbidden subgraphs for the torus found by Hlavacek coincides with the list of 9-vertex minimal forbidden subgraphs for the torus found by Myrvold [13].

The remainder of this paper is organized as follows. In Section 2, we find Kuratowski coverings for all minimal forbidden subgraphs of order < 10 for $\mathbb{N}_3$ and $\mathbb{N}_4$, which prove Theorem 1.1.

Remark 1. Theorem 1.1 is part of the main result of author's thesis [10], [9] : Every minimal forbidden subgraph of order < 10 satisfies the Kuratowski covering conjecture.

Remark 2. A strengthened form of the Kuratowski covering conjecture analogous to the complete Kuratowski theorem for the plane says that a finite graph G fails to embed in $\mathbb{N}_{\tilde{g}}$ if and only if there are $\tilde{g}+1$ Kuratowski subgraphs in G satisfying the conditions of the Kuratowski covering conjecture.

2. Kuratowski coverings for $\mathbb{N}_3$ and $\mathbb{N}_4$

2.1. **Kuratowski coverings for $\mathbb{N}_3$.** The complete list of minimal forbidden subgraphs of order < 10 for $\mathbb{N}_3$ is given in [8]. Since the genus of $\mathbb{N}_3$ is three, for each minimal forbidden subgraph G of order < 10 for $\mathbb{N}_3$, we find four Kuratowski subgraphs G_1, G_2, G_3, G_4 as a Kuratowski covering such that the union of every pair of these contains a subdivision of a minimal forbidden subgraph for the projective plane and the union of every triple of these contains a subdivision of a minimal forbidden subgraph for the torus and a subdivision of a minimal forbidden subgraph for the Klein bottle. We show these Kuratowski coverings in Figure 1, ... , 34. The names of minimal forbidden subgraphs for the projective are from [5]. The 8-vertex minimal forbidden subgraphs for the torus are $K_8 - K_3$, $K_8 - (K_{1,2} \cup 2K_2)$, and $K_8 - K_{2,3}$ [4], and the 8-vertex minimal forbidden subgraphs for the Klein bottle are $K_8 - 4K_2$, $K_8 - (K_3 \vee K_2)$, $K_8 - 2K_3$, $K_8 - 2K_{1,3}$, and $K_8 - (K_{1,4} \cup K_3)$, which are from [7]. The names of 9-vertex minimal forbidden subgraphs for the torus are from [6] and there are 63 9-vertex minimal forbidden subgraphs $\tilde{I}^2_{9,1}, \ldots, \tilde{I}^2_{9,63}$ for the Klein bottle [3], [10].

2.2. **Kuratowski coverings for $\mathbb{N}_4$.** There are two minimal forbidden subgraphs $K_9 - K_{1,2}$ and $K_9 - 2K_2$ of order < 10 for $\mathbb{N}_4$ [8] and we need five Kuratowski subgraphs G_1, G_2, G_3, G_4, G_5 as a Kuratowski covering which satisfies the conditions

of the Kuratowski covering conjecture. We show Kuratowski coverings of $K_9 - K_{1,2}$ and $K_9 - 2K_2$ in Figure 35, ... , 40.

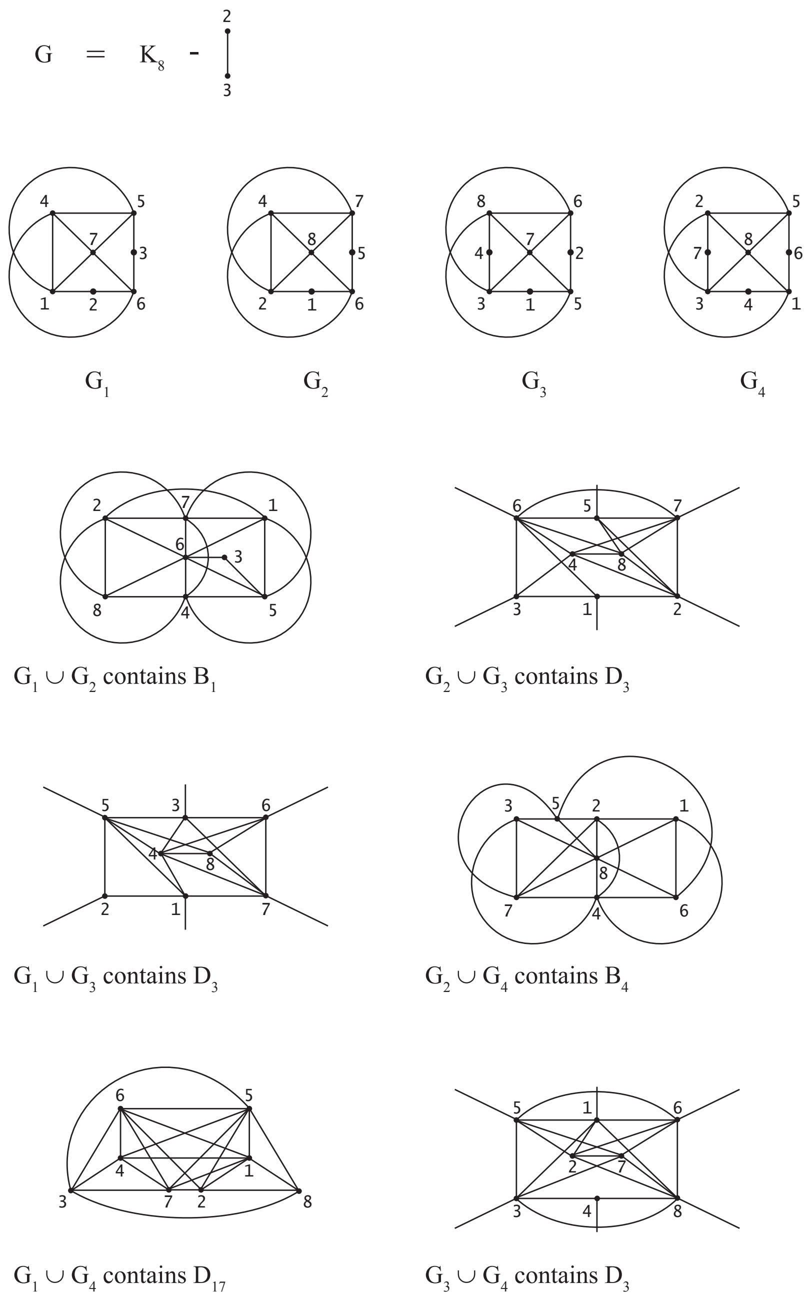
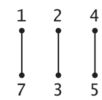
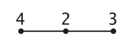


FIGURE 1. Kuratowski covering of $G = \tilde{I}^3_{8,1}$ for $\mathbb{N}_3$

$G_2 \cup G_3 \cup G_4$ contains
K_8 - $(K_{1,2} \cup 2K_2)$ and K_8 - $4K_2$

K_8 -


$G_1 \cup G_3 \cup G_4$ contains
K_8 - $(K_{1,2} \cup 2K_2)$ and K_8 - $(K_3 \vee K_2)$

K_8 -


$G_1 \cup G_2 \cup G_4$ is isomorphic
to $G_1 \cup G_3 \cup G_4$

K_8 -

$G_1 \cup G_2 \cup G_3$ contains
K_8 - $K_{2,3}$ and K_8 - $(K_3 \vee K_2)$

K_8 -

FIGURE 2. Kuratowski covering of $G = \tilde{I}^3_{8,1}$ for $\mathbb{N}_3$ (Continued)

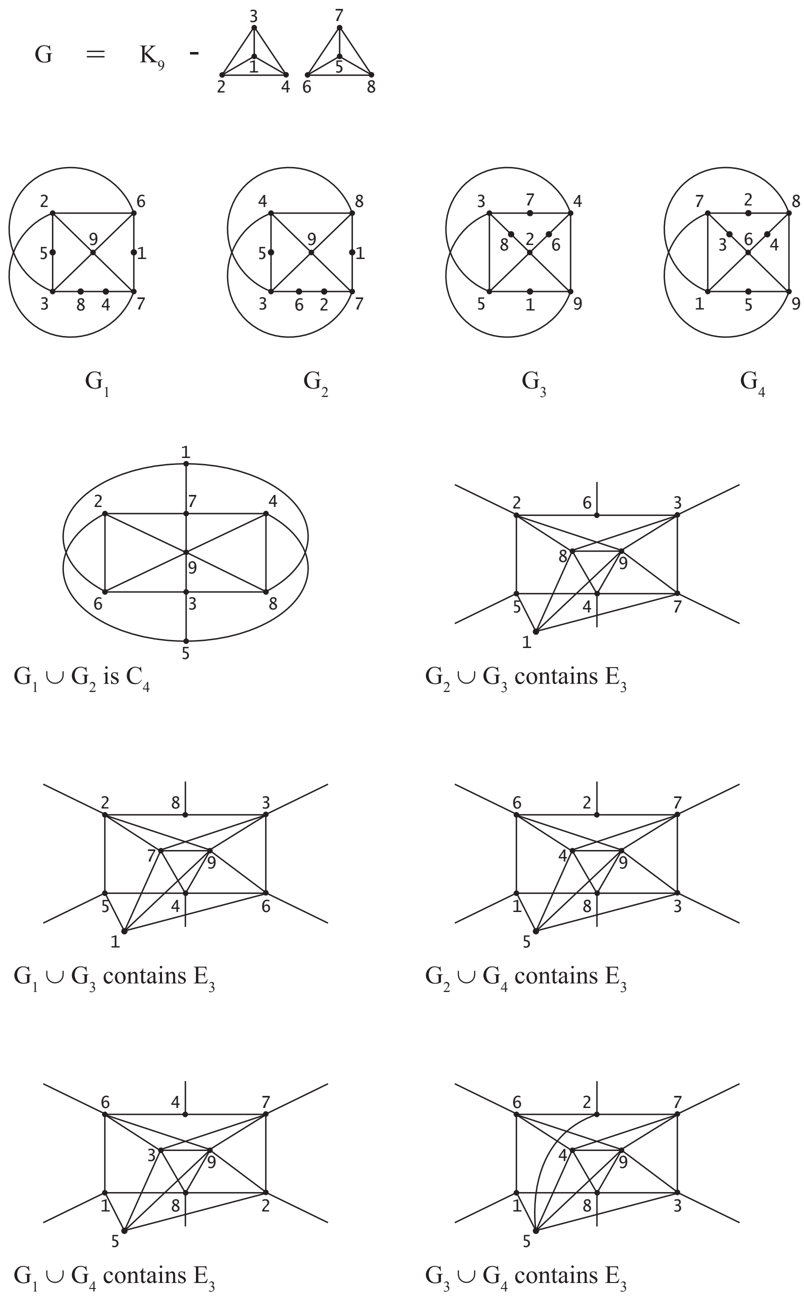
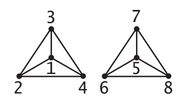
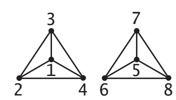


FIGURE 3. Kuratowski covering of $G = \tilde{I}^3_{9,1}$ for $\mathbb{N}_3$

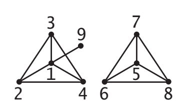
$G_2 \cup G_3 \cup G_4$ contains S5.5 and $\tilde{I}^2_{9,38}$

K_9 -


$G_1 \cup G_3 \cup G_4$ contains S5.5 and $\tilde{I}^2_{9,38}$

K_9 -


$G_1 \cup G_2 \cup G_4$ contains S5.5 and $\tilde{I}^2_{9,38}$

K_9 -


$G_1 \cup G_2 \cup G_3$ contains S5.5 and $\tilde{I}^2_{9,38}$

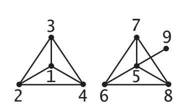
K_9 -


FIGURE 4. Kuratowski covering of $G = \tilde{I}^3_{9,1}$ for $\mathbb{N}_3$ (Continued)

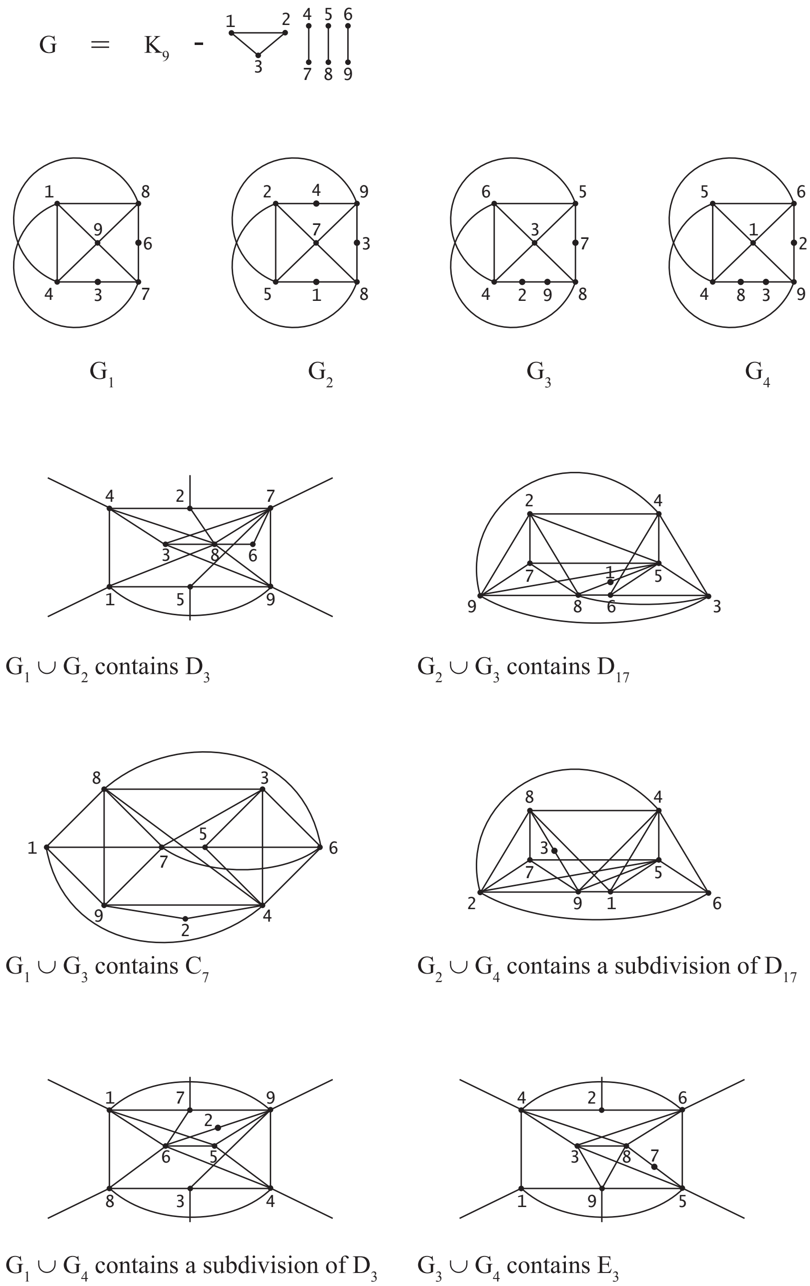


FIGURE 5. Kuratowski covering of $G = \tilde{I}^3_{9,2}$ for $\mathbb{N}_3$

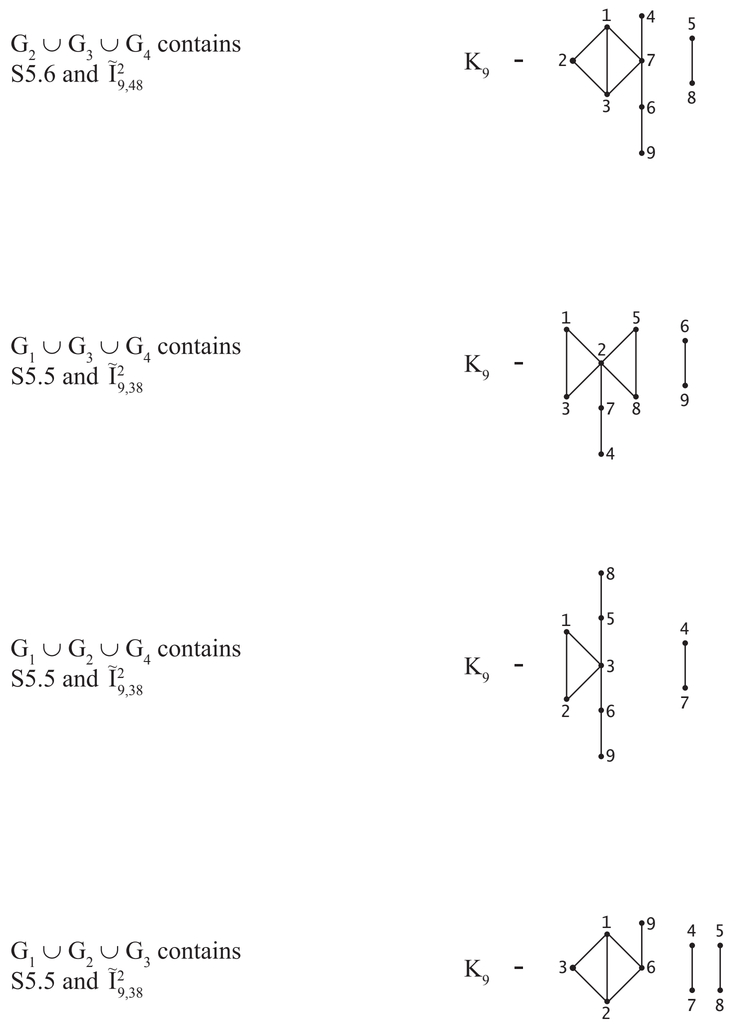


FIGURE 6. Kuratowski covering of $G = \tilde{I}^3_{9,2}$ for $\mathbb{N}_3$ (Continued)

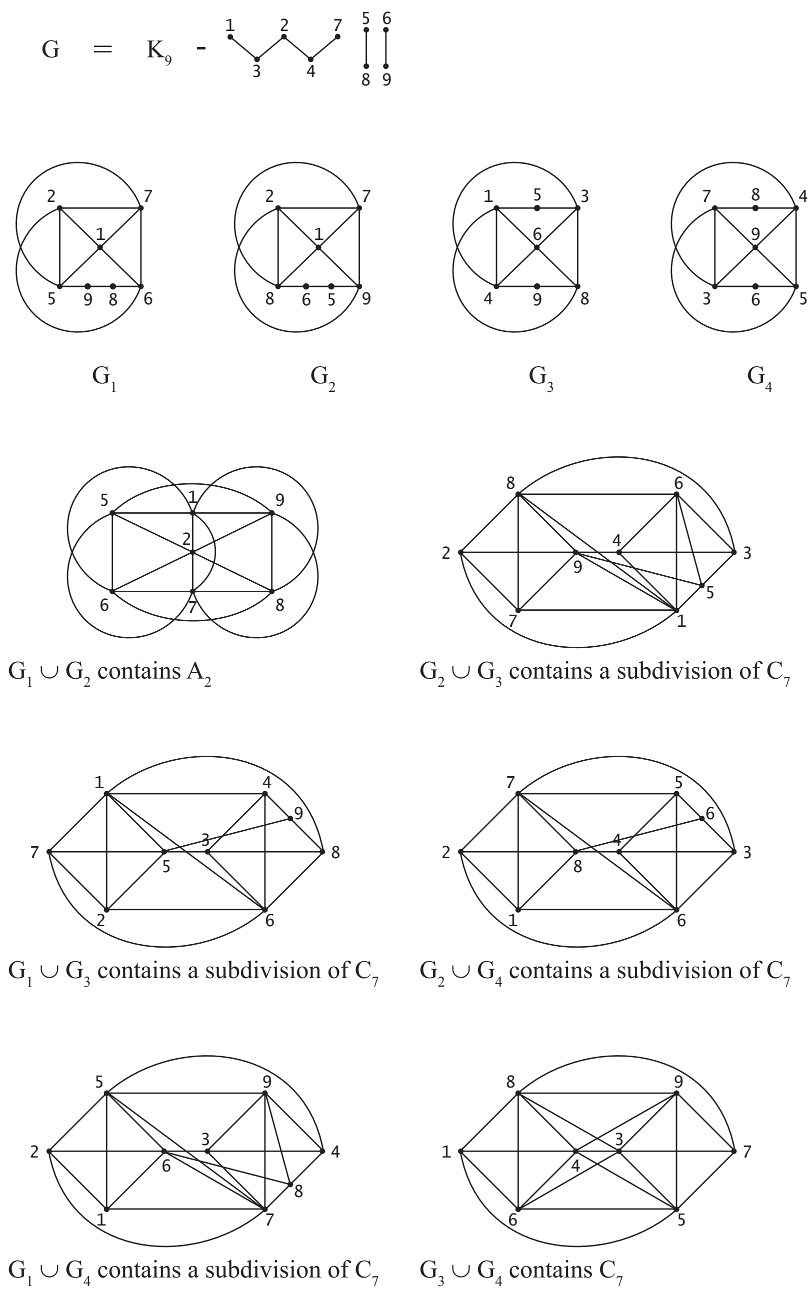


FIGURE 7. Kuratowski covering of $G = \tilde{I}^3_{9,3}$ for $\mathbb{N}_3$

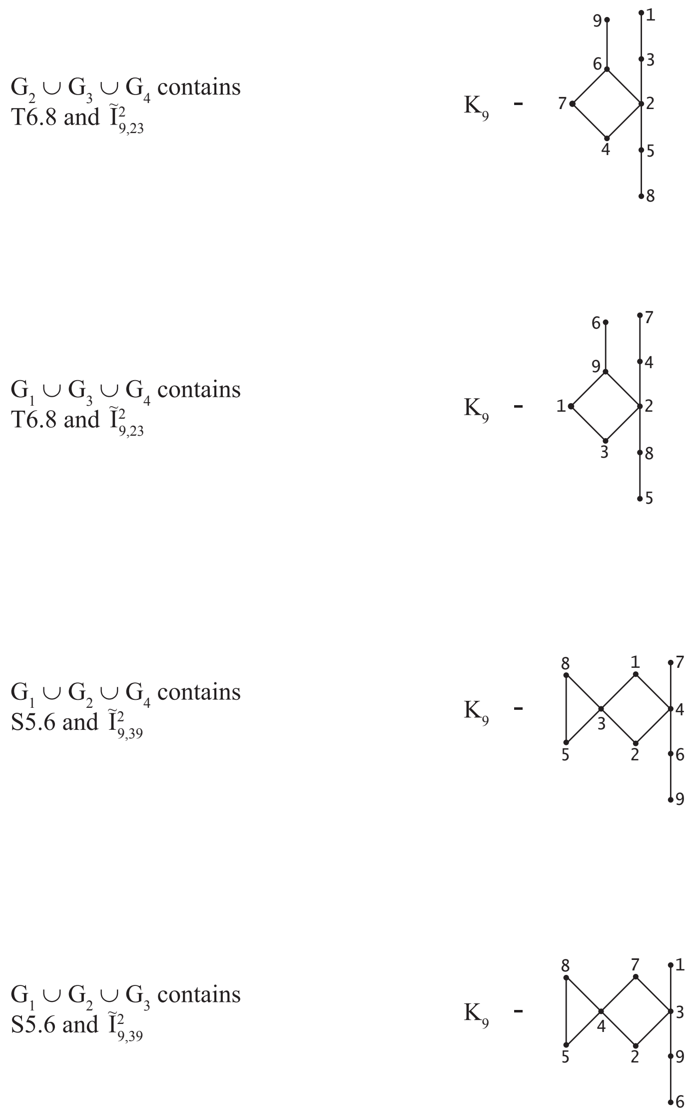


FIGURE 8. Kuratowski covering of $G = \tilde{I}^3_{9,3}$ for $\mathbb{N}_3$ (Continued)

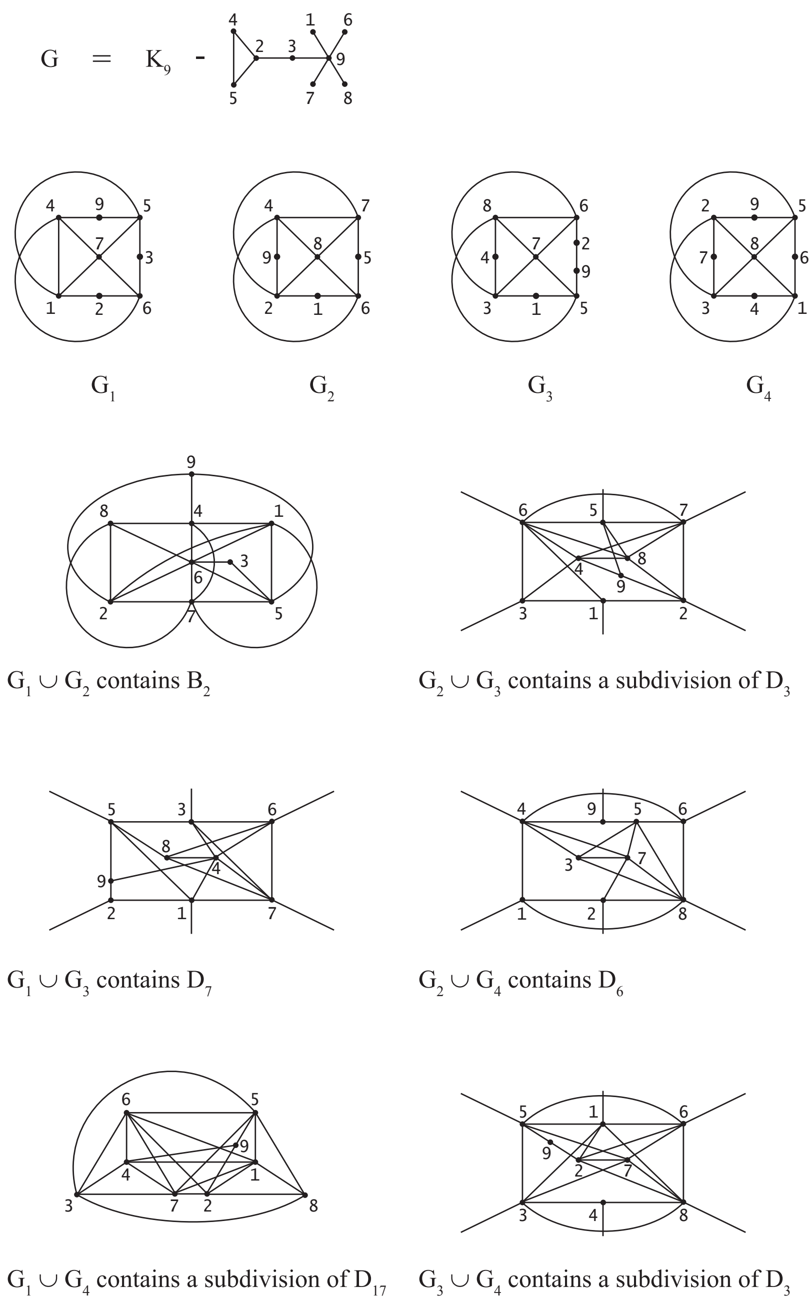
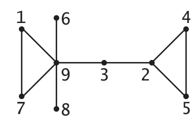


FIGURE 9. Kuratowski covering of $G = \tilde{I}^3_{9,4}$ for $\mathbb{N}_3$

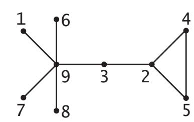
$G_2 \cup G_3 \cup G_4$ contains S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



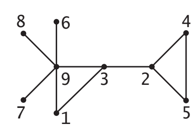
$G_1 \cup G_3 \cup G_4$ contains S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains U6.6a and $\tilde{I}^2_{9,42}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains U6.6a and $\tilde{I}^2_{9,42}$

K_9 -

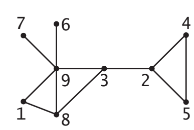


FIGURE 10. Kuratowski covering of $G = \tilde{I}^3_{9,4}$ for $\mathbb{N}_3$ (Continued)

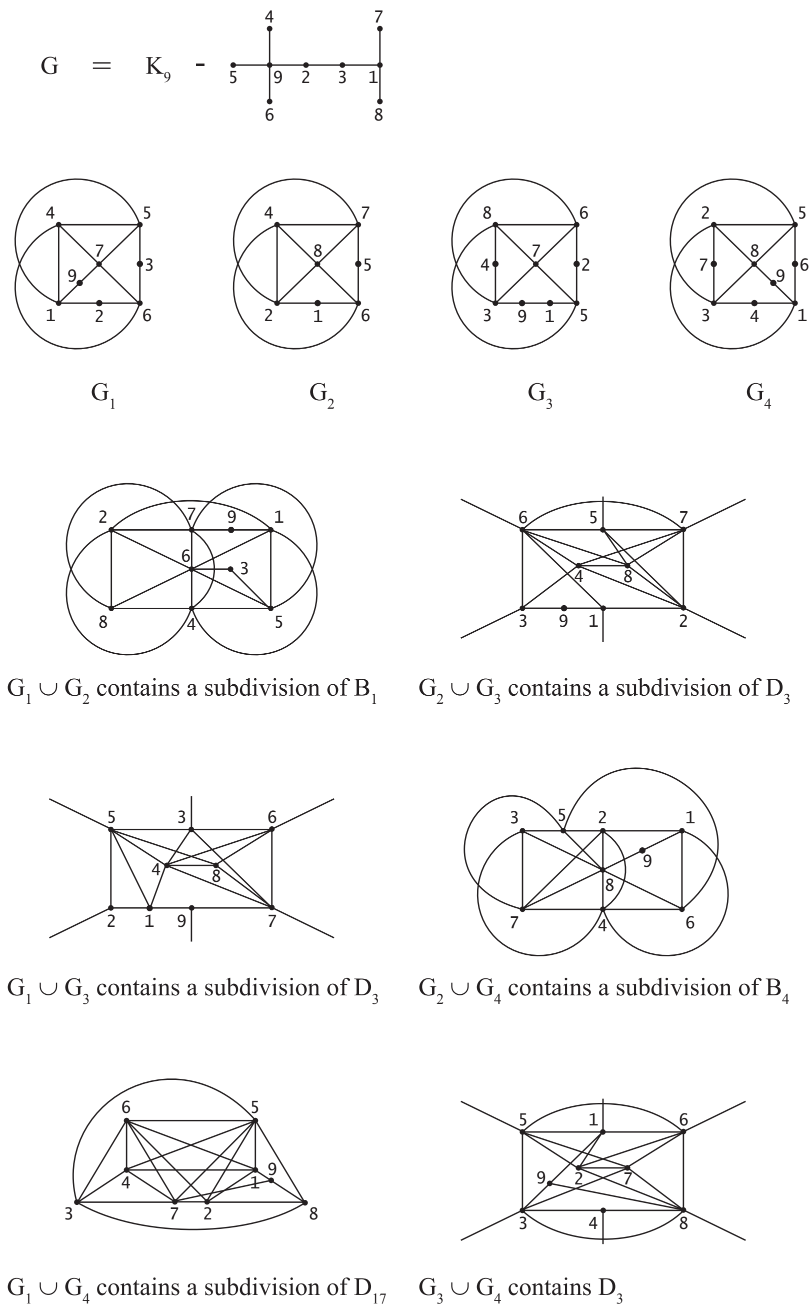
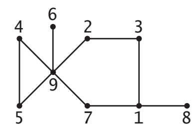


FIGURE 11. Kuratowski covering of $G = \tilde{I}^3_{9,5}$ for $\mathbb{N}_3$

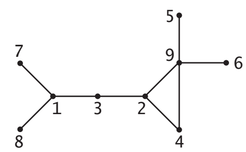
$G_2 \cup G_3 \cup G_4$ contains S5.6 and $\tilde{I}^2_{9,37}$

K_9 -



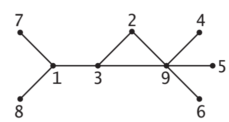
$G_1 \cup G_3 \cup G_4$ contains S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains V6.5 and $\tilde{I}^2_{9,39}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains U6.6a and $\tilde{I}^2_{9,41}$

K_9 -

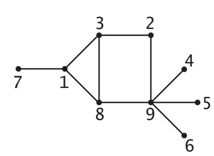


FIGURE 12. Kuratowski covering of $G = \tilde{I}^3_{9,5}$ for $\mathbb{N}_3$ (Continued)

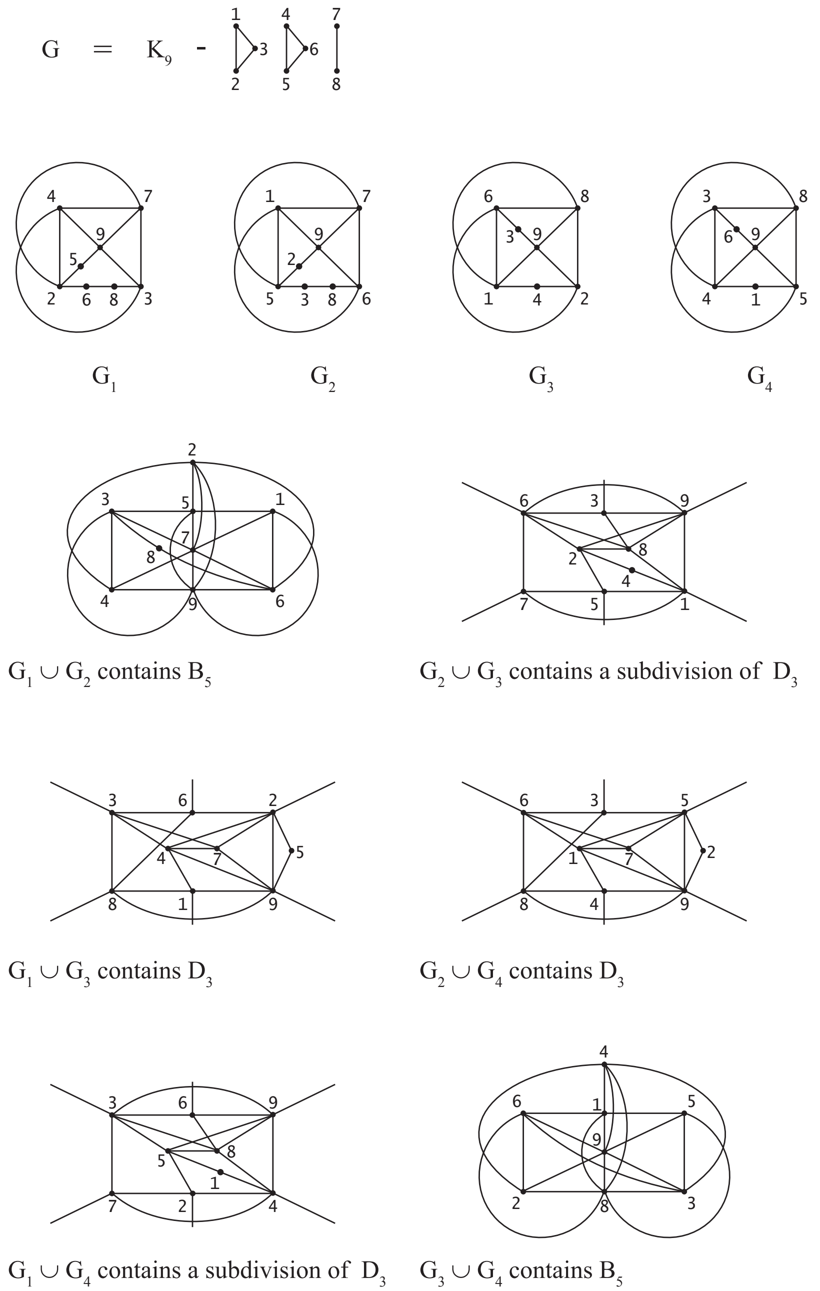
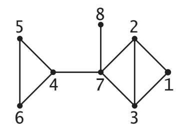


FIGURE 13. Kuratowski covering of $G = \tilde{I}^3_{9,6}$ for $\mathbb{N}_3$

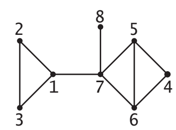
$G_2 \cup G_3 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



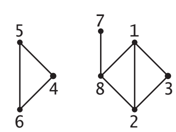
$G_1 \cup G_3 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -

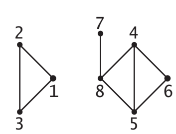


FIGURE 14. Kuratowski covering of $G = \tilde{I}^3_{9,6}$ for $\mathbb{N}_3$ (Continued)

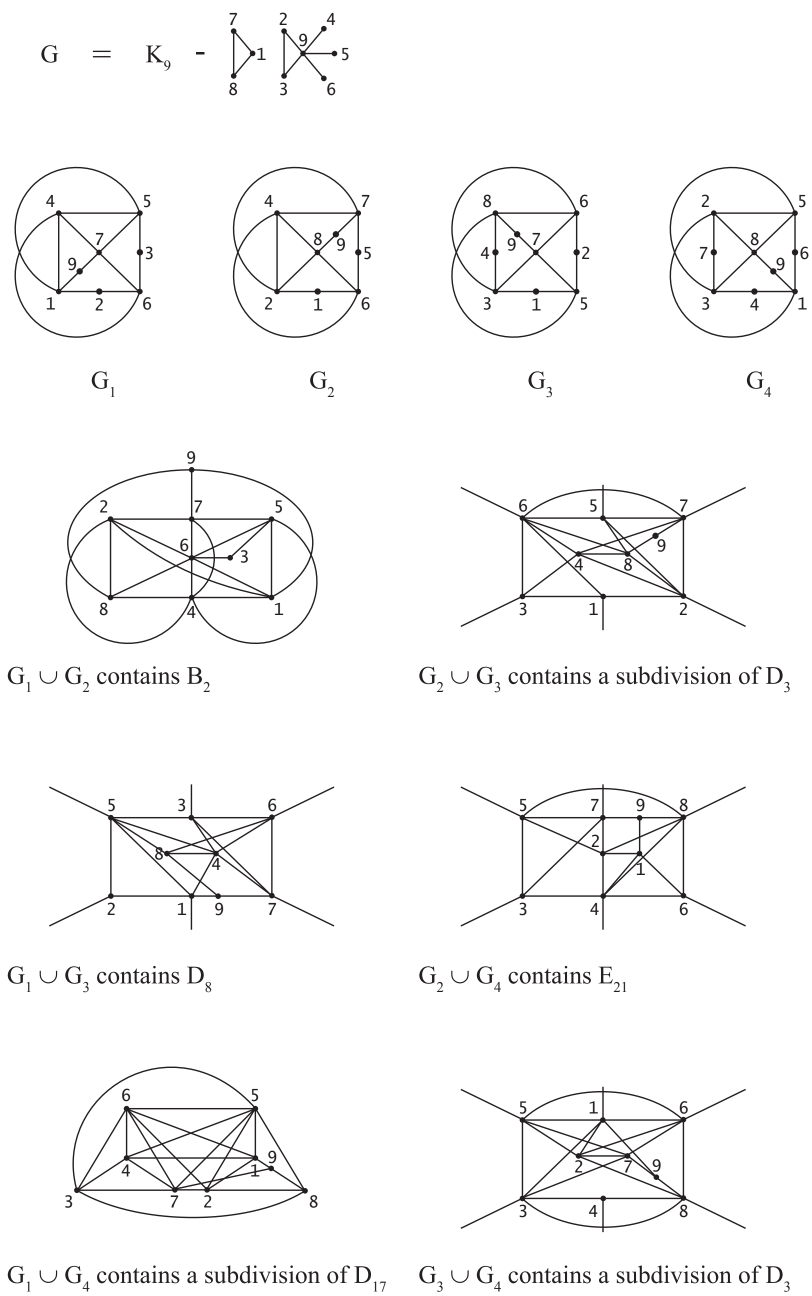
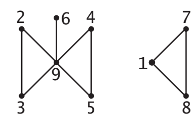


FIGURE 15. Kuratowski covering of $G = \tilde{I}^3_{9,7}$ for $\mathbb{N}_3$

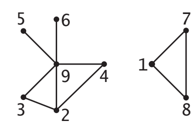
$G_2 \cup G_3 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



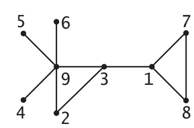
$G_1 \cup G_3 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains
U6.6a and $\tilde{I}^2_{9,42}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains
U6.6a and $\tilde{I}^2_{9,42}$

K_9 -

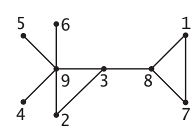


FIGURE 16. Kuratowski covering of $G = \tilde{I}^3_{9,7}$ for $\mathbb{N}_3$ (Continued)

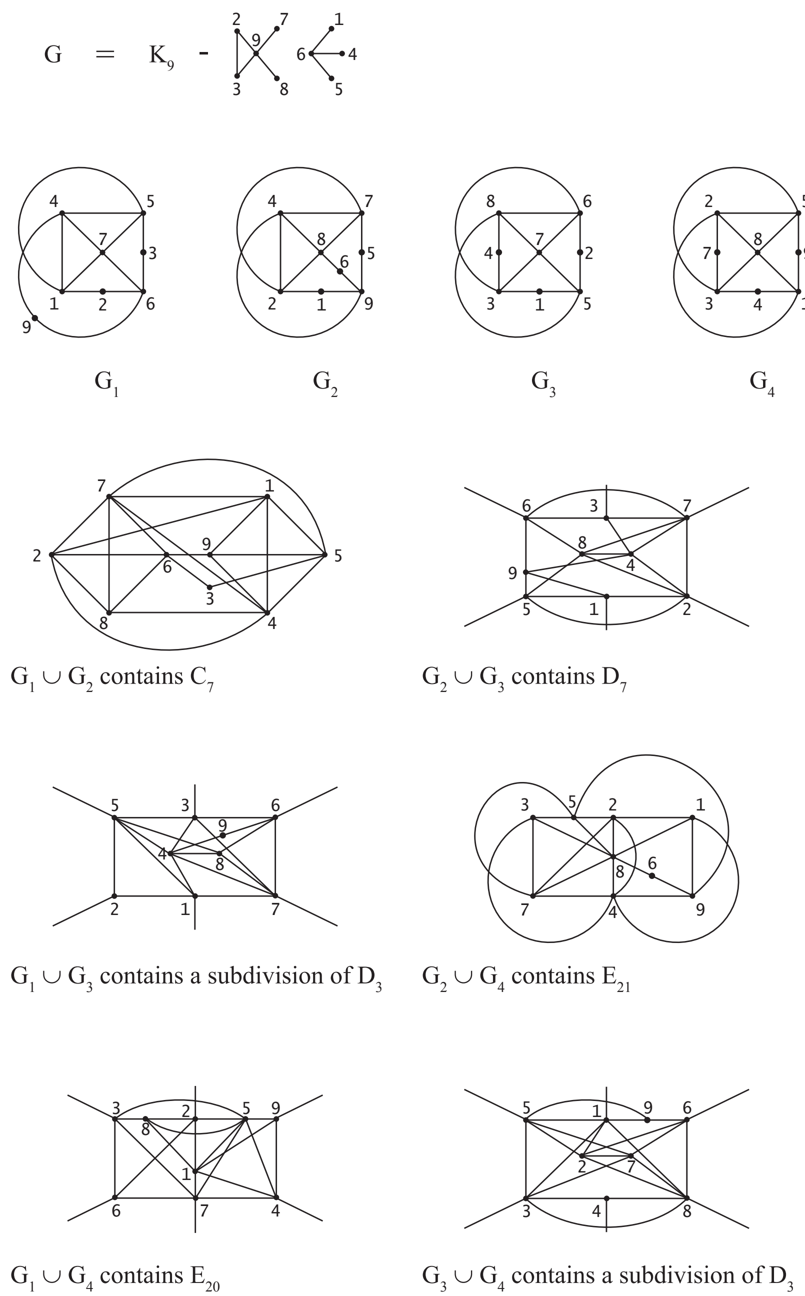
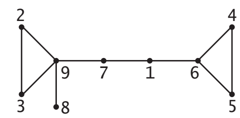


FIGURE 17. Kuratowski covering of $G = \tilde{I}^3_{9,8}$ for $\mathbb{N}_3$

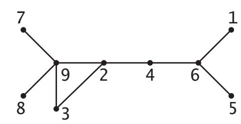
$G_2 \cup G_3 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



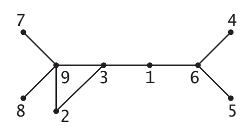
$G_1 \cup G_3 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -

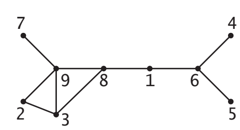


FIGURE 18. Kuratowski covering of $G = \tilde{I}^3_{9,8}$ for $\mathbb{N}_3$ (Continued)

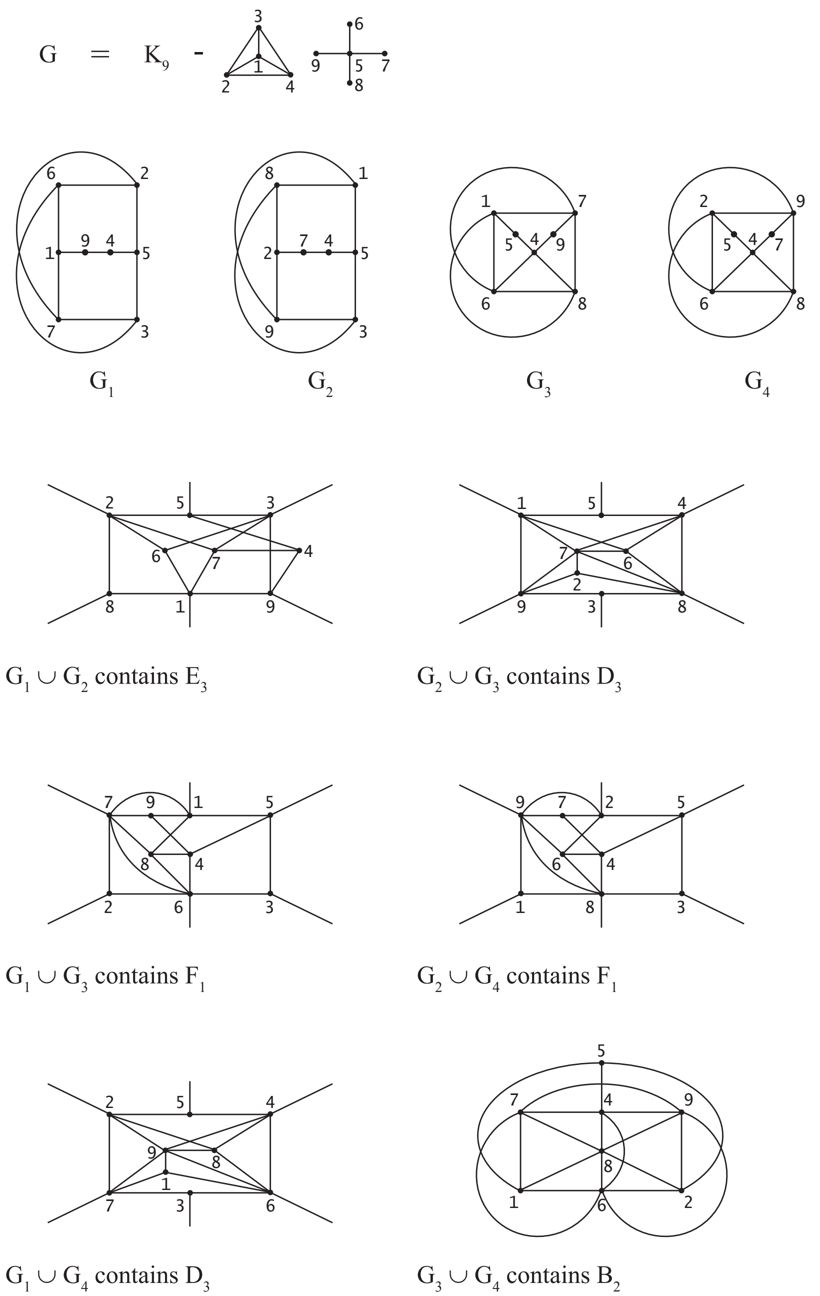
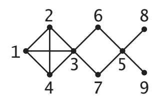


Figure 19. Kuratowski covering of $G = \tilde{I}^3_{9,9}$ for $\mathbb{N}_3$

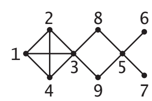
$G_2 \cup G_3 \cup G_4$ contains S5.6 and $\tilde{I}^2_{9,50}$

K_9 -



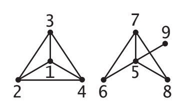
$G_1 \cup G_3 \cup G_4$ contains S5.6 and $\tilde{I}^2_{9,50}$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains S5.5 and $\tilde{I}^2_{9,38}$

K_9 -

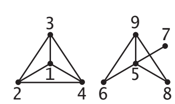


FIGURE 20. Kuratowski covering of $G = \tilde{I}^3_{9,9}$ for $\mathbb{N}_3$ (Continued)

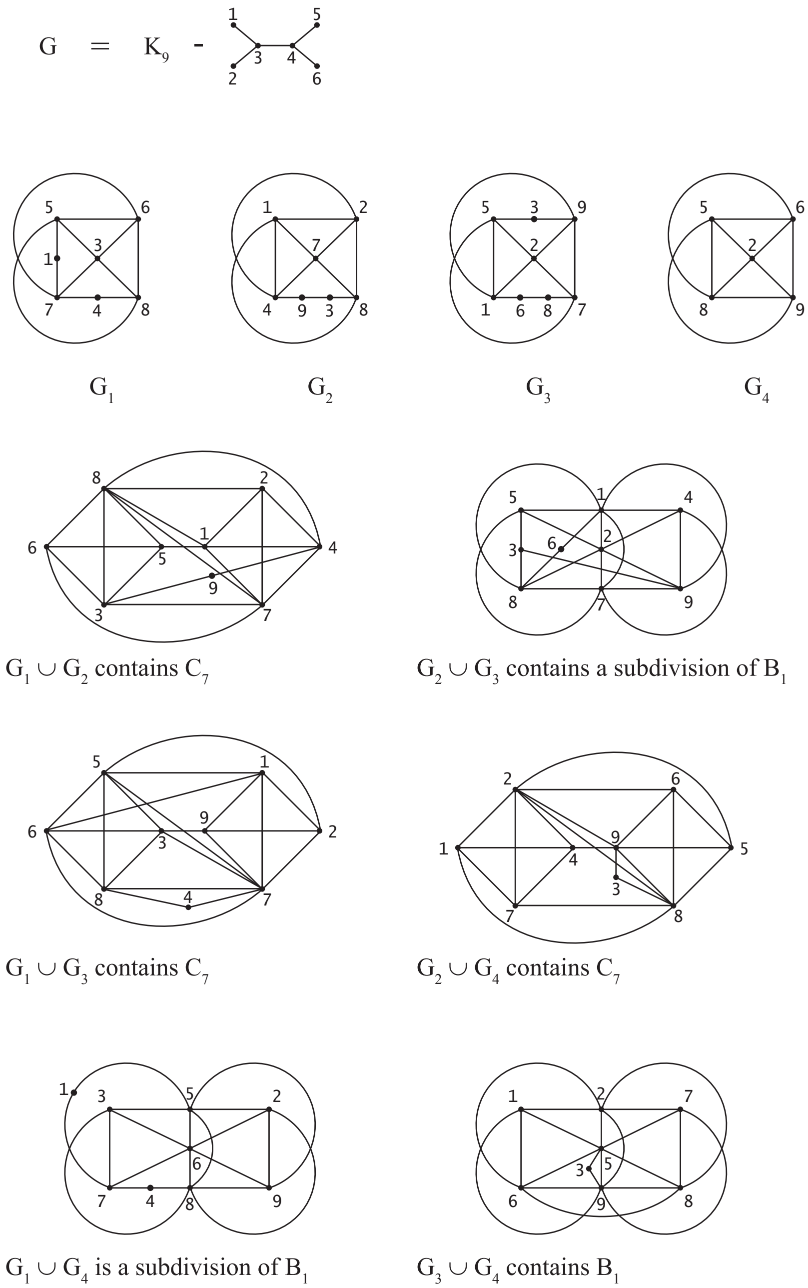
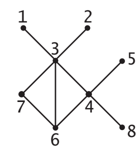


FIGURE 21. Kuratowski covering of $G = \tilde{I}^3_{9,10}$ for $\mathbb{N}_3$

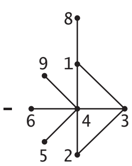
$G_2 \cup G_3 \cup G_4$ contains U6.6a and $\tilde{I}^2_{9,42}$

K_9 -



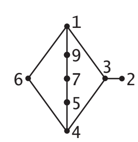
$G_1 \cup G_3 \cup G_4$ contains K_8 - $K_{2,3}$ and K_8 - $K_3 \vee K_2$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains T5.8 and $\tilde{I}^2_{9,29}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains T5.8 and $\tilde{I}^2_{9,24}$

K_9 -

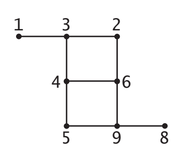


FIGURE 22. Kuratowski covering of $G = \tilde{I}^3_{9,10}$ for $\mathbb{N}_3$ (Continued)

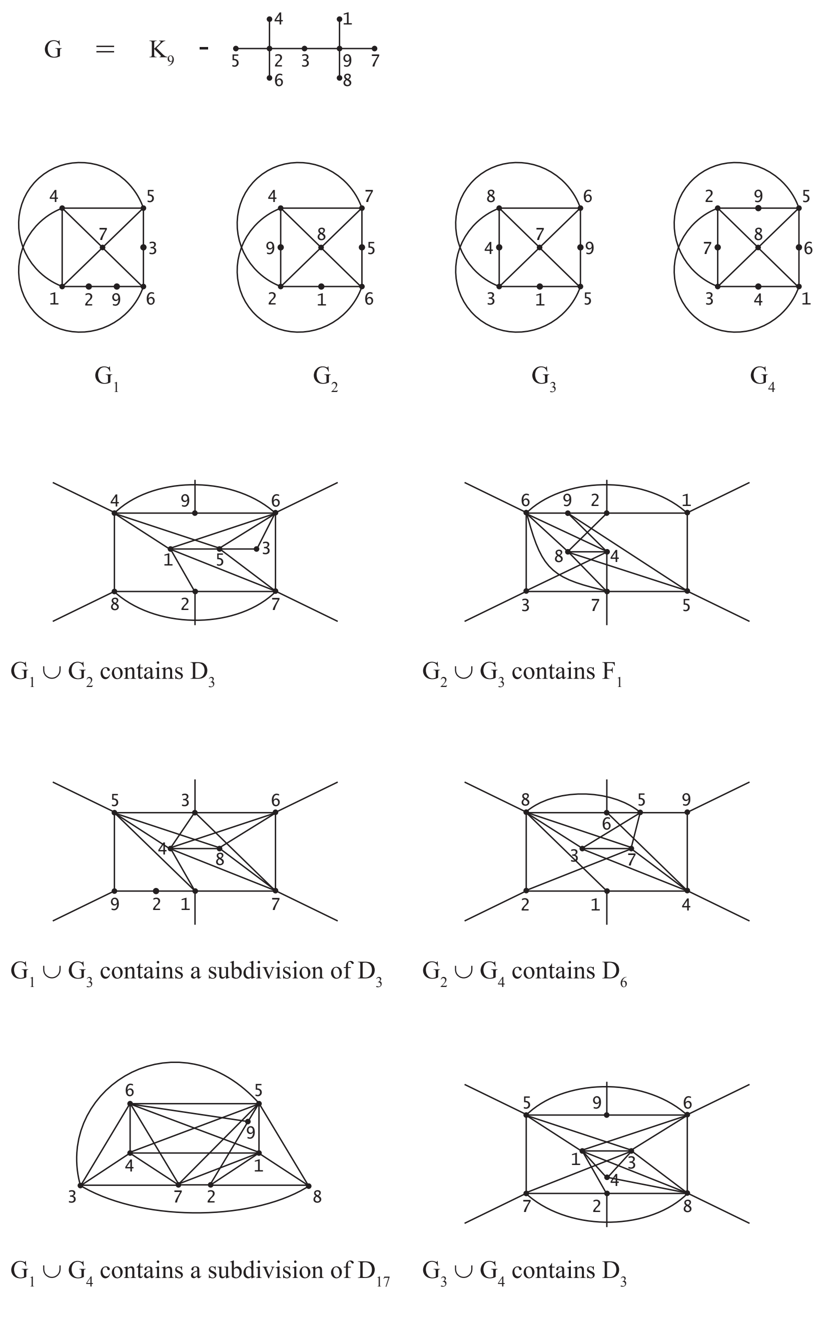
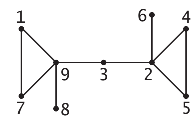


FIGURE 23. Kuratowski covering of $G = \tilde{I}^3_{9,11}$ for $\mathbb{N}_3$

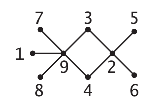
$G_2 \cup G_3 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



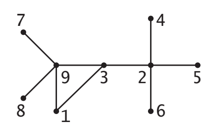
$G_1 \cup G_3 \cup G_4$ contains
S5.6 and $\tilde{I}^2_{9,39}$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -

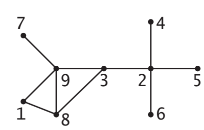


FIGURE 24. Kuratowski covering of $G = \tilde{I}^3_{9,11}$ for $\mathbb{N}_3$ (Continued)

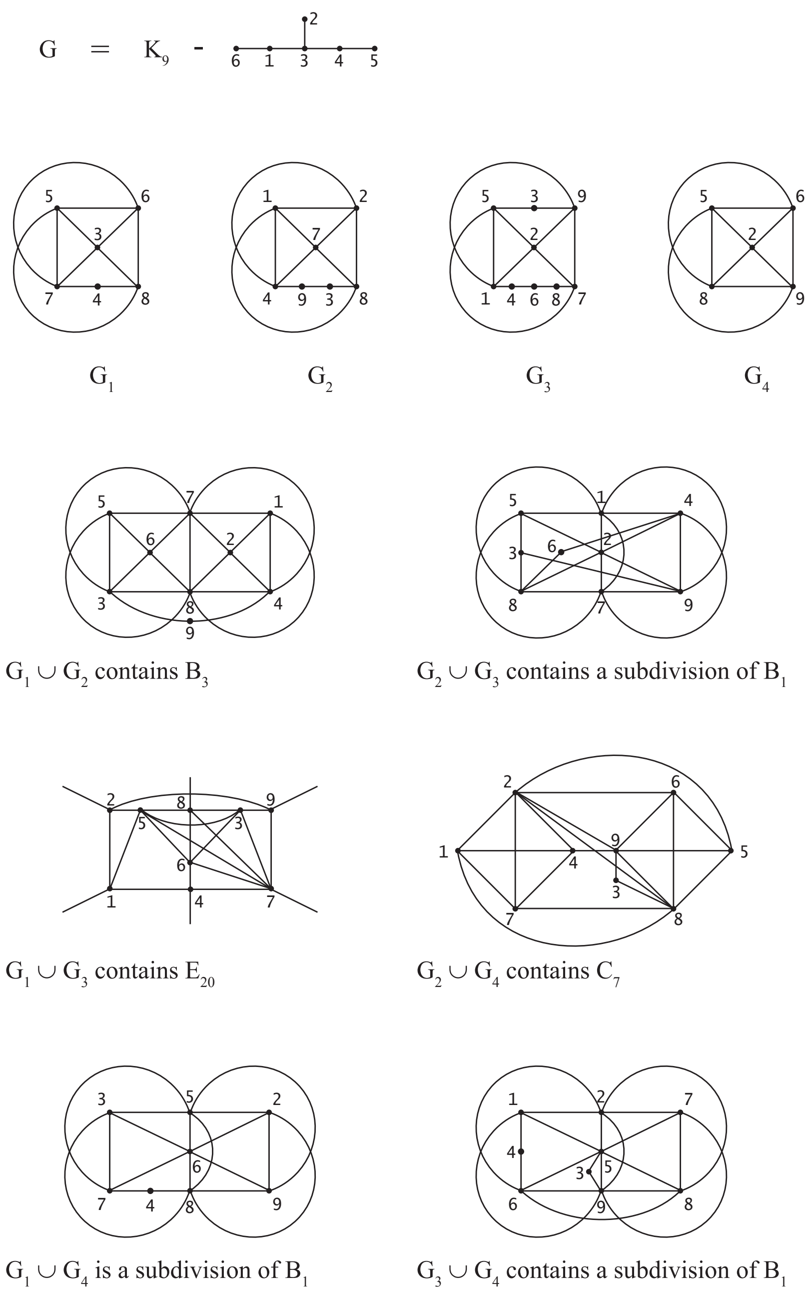
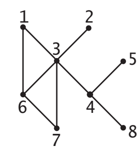


FIGURE 25. Kuratowski covering of $G = \tilde{I}^3_{9,12}$ for $\mathbb{N}_3$

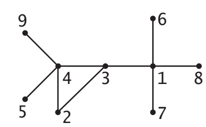
$G_2 \cup G_3 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



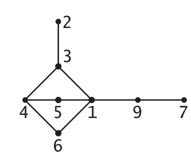
$G_1 \cup G_3 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains
T5.8 and $\tilde{I}^2_{9,27}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains
T5.8 and $\tilde{I}^2_{9,35}$

K_9 -

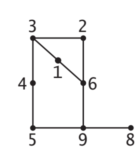


FIGURE 26. Kuratowski covering of $G = \tilde{I}^3_{9,12}$ for $\mathbb{N}_3$ (Continued)

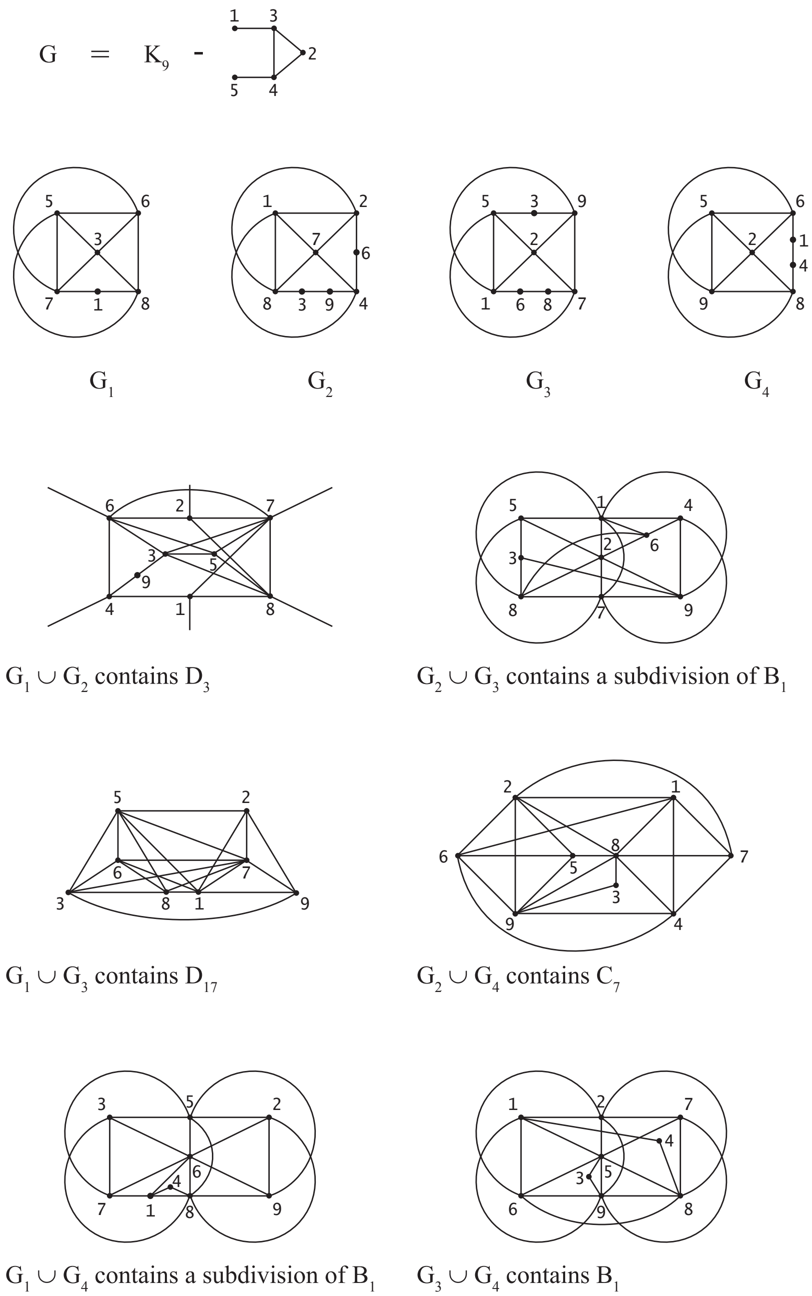


FIGURE 27. Kuratowski covering of $G = \tilde{I}^3_{9,13}$ for $\mathbb{N}_3$

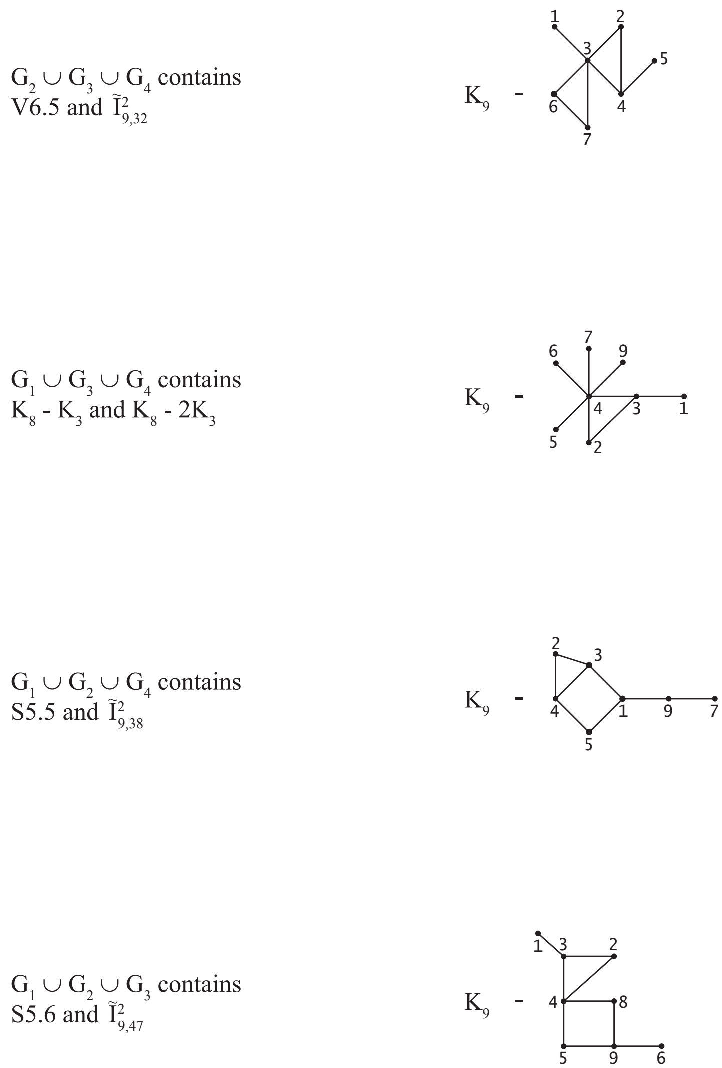


FIGURE 28. Kuratowski covering of $G = \tilde{I}^3_{9,13}$ for $\mathbb{N}_3$ (Continued)

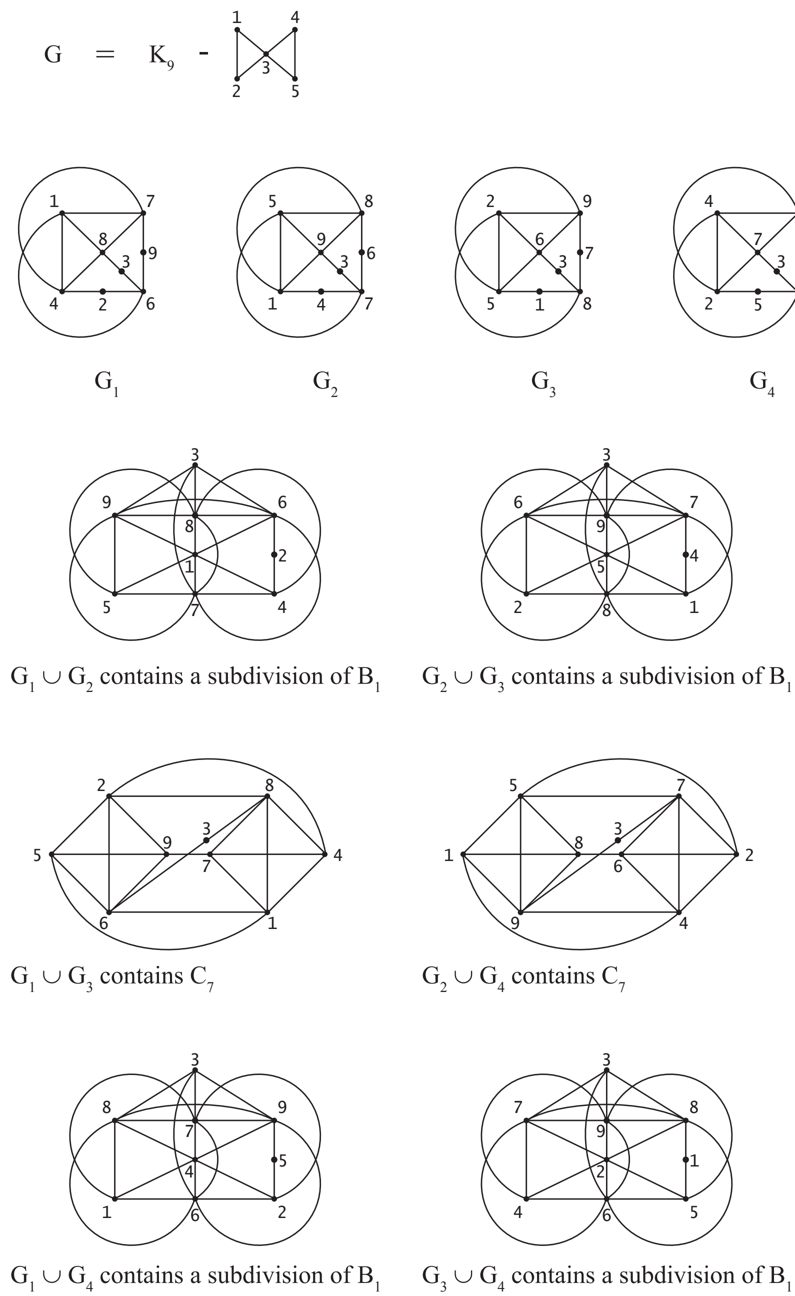
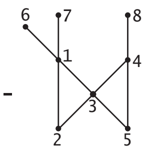


FIGURE 29. Kuratowski covering of $G = \tilde{I}^3_{9,14}$ for $\mathbb{N}_3$

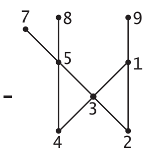
$G_2 \cup G_3 \cup G_4$ contains V6.4 and $\tilde{I}^2_{9,41}$

K_9 -



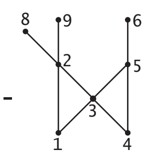
$G_1 \cup G_3 \cup G_4$ contains V6.4 and $\tilde{I}^2_{9,41}$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains V6.4 and $\tilde{I}^2_{9,41}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains V6.4 and $\tilde{I}^2_{9,41}$

K_9 -

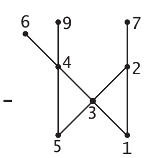
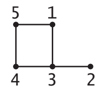
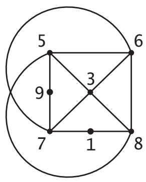


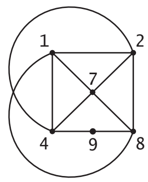
FIGURE 30. Kuratowski covering of $G = \tilde{I}^3_{9,14}$ for $\mathbb{N}_3$ (Continued)

$G = K_9$ -

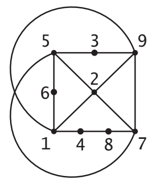




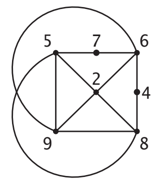
G_1



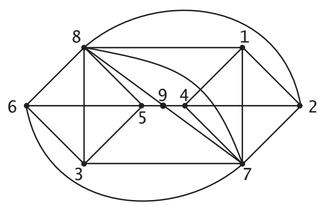
G_2



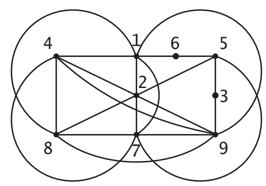
G_3



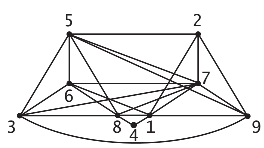
G_4



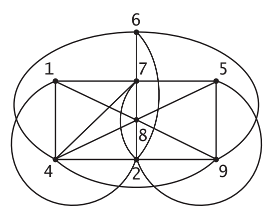
$G_1 \cup G_2$ contains a subdivision of C_7



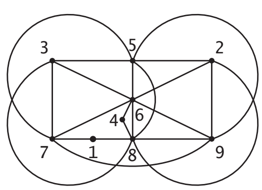
$G_2 \cup G_3$ contains a subdivision of B_1



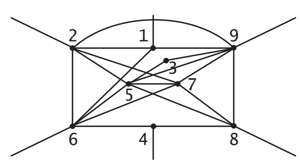
$G_1 \cup G_3$ contains D_{17}



$G_2 \cup G_4$ contains B_5



$G_1 \cup G_4$ contains a subdivision of B_1

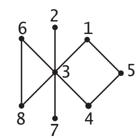


$G_3 \cup G_4$ contains D_3

FIGURE 31. Kuratowski covering of $G = \tilde{I}^3_{9,15}$ for $\mathbb{N}_3$

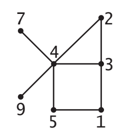
$G_2 \cup G_3 \cup G_4$ contains
$K_8 - (K_{1,2} \cup 2K_2)$ and $K_8 - 2K_3$

K_9 -



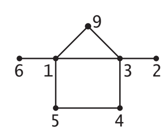
$G_1 \cup G_3 \cup G_4$ contains
U6.6a and $\tilde{I}^2_{9,39}$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains
V7.6 and $\tilde{I}^2_{9,6}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains
T5.8 and $\tilde{I}^2_{9,21}$

K_9 -

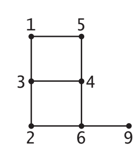
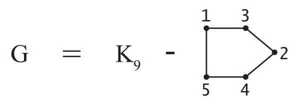
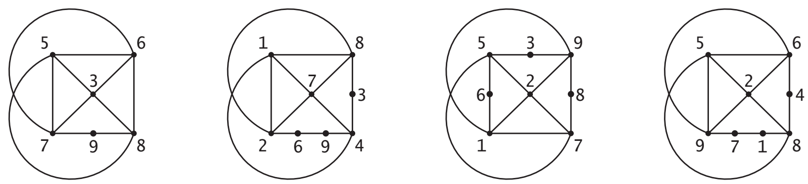
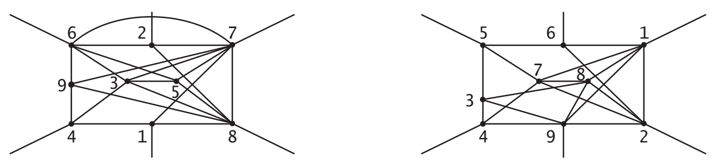


FIGURE 32. Kuratowski covering of $G = \tilde{I}^3_{9,15}$ for $\mathbb{N}_3$ (Continued)

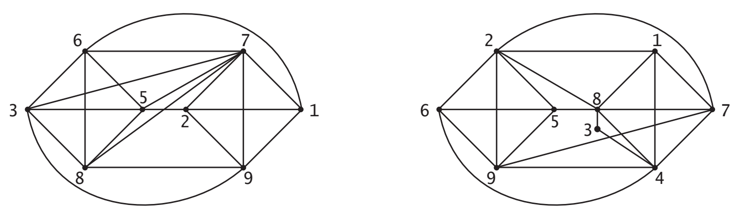




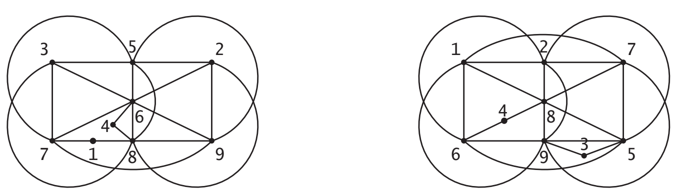
G_1 G_2 G_3 G_4



$G_1 \cup G_2$ contains a subdivision of D_3 $G_2 \cup G_3$ contains D_7



$G_1 \cup G_3$ contains C_7 $G_2 \cup G_4$ contains C_7

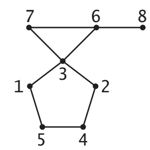


$G_1 \cup G_4$ contains a subdivision of B_1 $G_3 \cup G_4$ contains a subdivision of B_1

FIGURE 33. Kuratowski covering of $G = \tilde{I}^3_{9,16}$ for $\mathbb{N}_3$

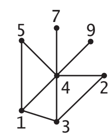
$G_2 \cup G_3 \cup G_4$ contains
S5.6 and $\tilde{I}^2_{9,5}$

K_9 -



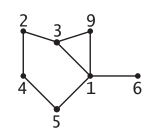
$G_1 \cup G_3 \cup G_4$ contains
K_8 - $K_{2,3}$ and K_8 - $(K_3 \vee K_2)$

K_9 -



$G_1 \cup G_2 \cup G_4$ contains
U6.6a and $\tilde{I}^2_{9,2}$

K_9 -



$G_1 \cup G_2 \cup G_3$ contains
S5.6 and $\tilde{I}^2_{9,6}$

K_9 -

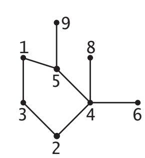


FIGURE 34. Kuratowski covering of $G = \tilde{I}^3_{9,16}$ for $\mathbb{N}_3$ (Continued)

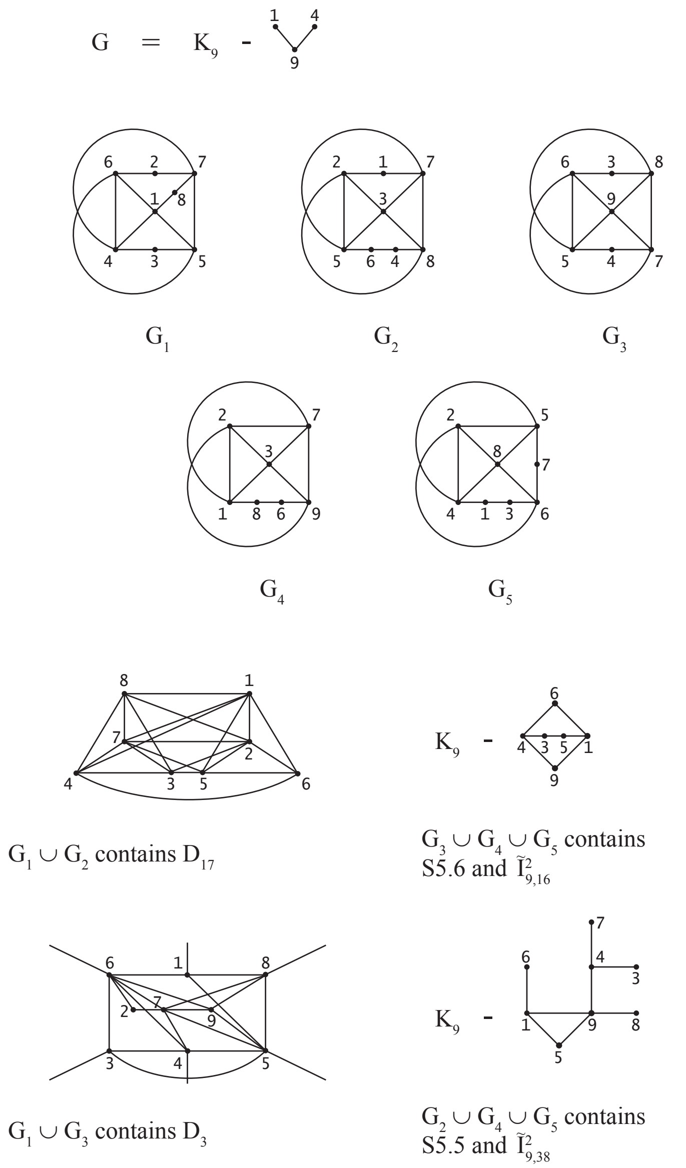
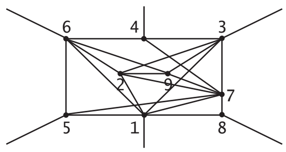
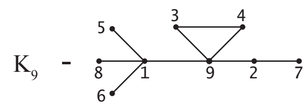


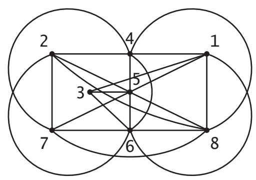
FIGURE 35. Kuratowski covering of $G = \tilde{I}^4_{9,1}$ for $\mathbb{N}_4$



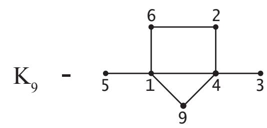
$G_1 \cup G_4$ contains D_5



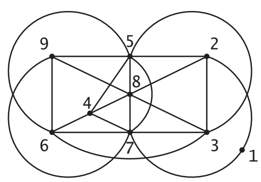
$G_2 \cup G_3 \cup G_5$ contains S5.5 and $\tilde{I}^2_{9,38}$



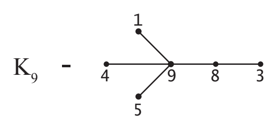
$G_1 \cup G_5$ contains B_1



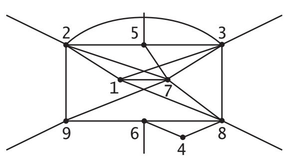
$G_2 \cup G_3 \cup G_4$ contains V7.7 and $\tilde{I}^2_{9,2}$



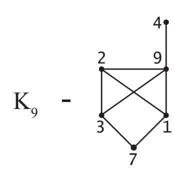
$G_2 \cup G_3$ contains a subdivision of B_1



$G_1 \cup G_4 \cup G_5$ contains S5.5 and $\tilde{I}^2_{9,38}$

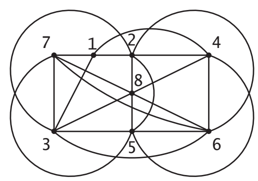


$G_2 \cup G_4$ contains D_3



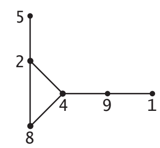
$G_1 \cup G_3 \cup G_5$ contains S5.5 and $\tilde{I}^2_{9,38}$

FIGURE 36. Kuratowski covering of $G = \tilde{I}^4_{9,1}$ for $\mathbb{N}_4$ (Continued)

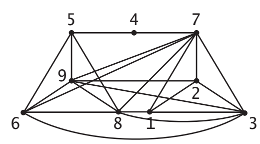


$G_2 \cup G_5$ contains a subdivision of B_1

K_9 -

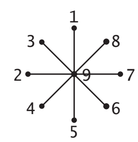


$G_1 \cup G_3 \cup G_4$ contains S5.5 and $\tilde{I}^2_{9,38}$

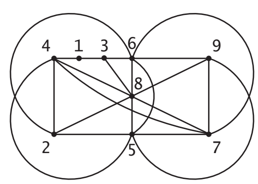


$G_3 \cup G_4$ contains a subdivision of D_{17}

K_9 -

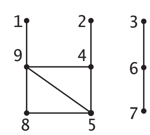


$G_1 \cup G_2 \cup G_5$ contains K_8 - $K_{2,3}$ and K_8 - $2K_{1,3}$

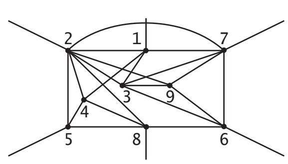


$G_3 \cup G_5$ contains a subdivision of B_1

K_9 -

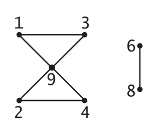


$G_1 \cup G_2 \cup G_4$ contains S5.5 and $\tilde{I}^2_{9,38}$



$G_4 \cup G_5$ contains D_3

K_9 -



$G_1 \cup G_2 \cup G_3$ contains S5.5 and $\tilde{I}^2_{9,38}$

FIGURE 37. Kuratowski covering of $G = \tilde{I}^4_{9,1}$ for $\mathbb{N}_4$ (Continued)

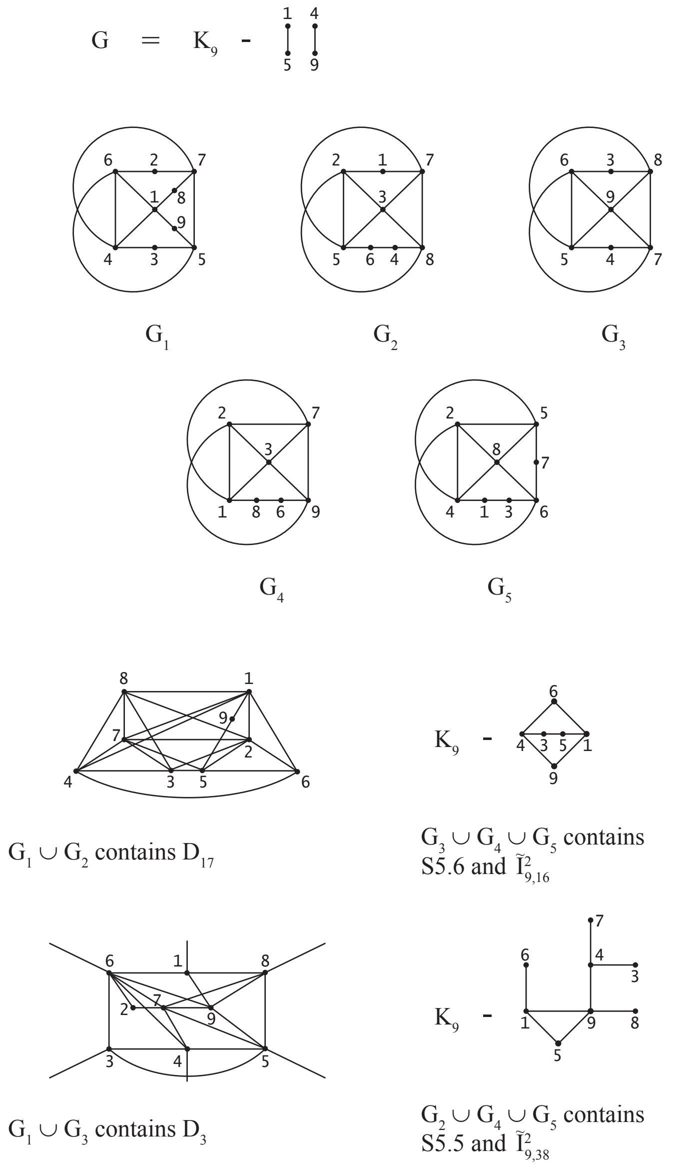
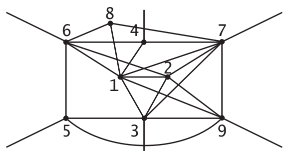
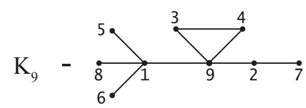


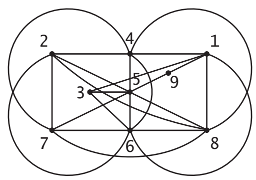
FIGURE 38. Kuratowski covering of $G = \tilde{I}^4_{9,2}$ for $\mathbb{N}_4$



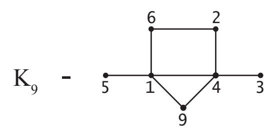
$G_1 \cup G_4$ contains D_3



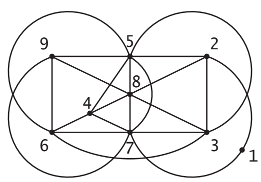
$G_2 \cup G_3 \cup G_5$ contains
S5.5 and $\tilde{I}^2_{9,38}$



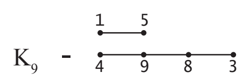
$G_1 \cup G_5$ contains a subdivision of B_1



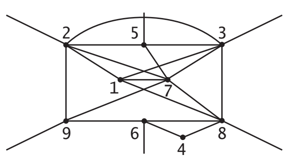
$G_2 \cup G_3 \cup G_4$ contains
V7.7 and $\tilde{I}^2_{9,2}$



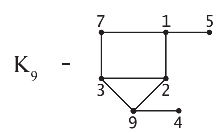
$G_2 \cup G_3$ contains a subdivision of B_1



$G_1 \cup G_4 \cup G_5$ contains
S5.5 and $\tilde{I}^2_{9,38}$

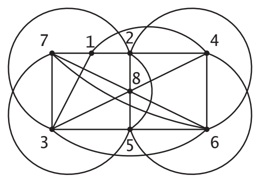


$G_2 \cup G_4$ contains D_3

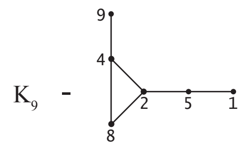


$G_1 \cup G_3 \cup G_5$ contains
S5.6 and $\tilde{I}^2_{9,8}$

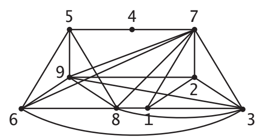
FIGURE 39. Kuratowski covering of $G = \tilde{I}^4_{9,2}$ for $\mathbb{N}_4$ (Continued)



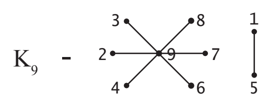
$G_2 \cup G_5$ contains a subdivision of B_1



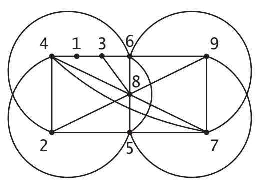
$G_1 \cup G_3 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$



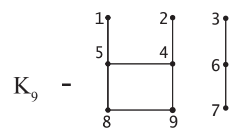
$G_3 \cup G_4$ contains a subdivision of D_{17}



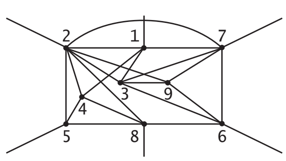
$G_1 \cup G_2 \cup G_5$ contains
K_8 - $K_{2,3}$ and K_8 - $2K_{1,3}$



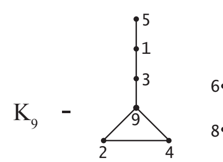
$G_3 \cup G_5$ contains a subdivision of B_1



$G_1 \cup G_2 \cup G_4$ contains
S5.5 and $\tilde{I}^2_{9,38}$



$G_4 \cup G_5$ contains D_3



$G_1 \cup G_2 \cup G_3$ contains
S5.5 and $\tilde{I}^2_{9,38}$

FIGURE 40. Kuratowski covering of $G = \tilde{I}^4_{9,2}$ for $\mathbb{N}_4$ (Continued)

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